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Long-range semi-infinite models with critical behaviour controlled by a dynamical system

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Abstract. The surface critical behaviour of the semi-infinite ferromagnetic n-vector Kac-Baker models is obtained. Near the ordinary and special transition points the self-consistency equations (which couple all the unknowns) are shown to be a small perturbation of a two-step recursion relation. The critical behaviour is thus shown to be controlled by the behaviour of a dynamical system near the bifurcation of a new hyperbolic fixed point. This provides both the local critical behaviour and the scaled magnetisation profile. Thereby, the range of validity of the continuum approximation is established.

1. Introduction

Many problems in condensed-matter physics involve the study of a sample with local average properties which vary along one direction, while in the other directions the sample is homogeneous. The study of such systems has been the subject of a large variety of papers, both theoretical and experimental. Mean-field theory is, as usual, a useful first approximation, since it often provides a correct picture of phase diagrams. However, in contrast with the simplicity of bulk mean-field theory, its inhomogeneous counterpart for lattice systems turns out to be rather complicated as it involves an infinite number of mean fields to be determined from a coupled system of non-linear equations.

Usually, the mean-field equations for inhomogeneous discrete systems appear as two-step recursion relations in which case the problem becomes one of non-linear dynamics: every solution is generated by a trajectory (subject to appropriate initial conditions) of a certain area-preserving map. This point of view has recently been put forward in the case of spin systems by Angelescu *et al* (1981b), Bak (1981) and Pandit and Wortis (1982). The method has been applied to the description of commensurate-incommensurate transitions (Bak 1981, Aubry 1983), surface critical behaviour and interfaces (Angelescu *et al* 1981a, b, 1983, Pandit and Wortis 1982) as well as adsorption phenomena (Pandit *et al* 1982). Numerical studies have shown that the relevant map has in general a large variety of different types of orbits. It is highly non-trivial to find among these the orbit corresponding to lowest free energy. The energetically stable states correspond to trajectories which are unstable with respect to small perturbations and this makes the numerical study extremely delicate. Under these circumstances, any analytical information is welcome. For free-surface ferromagnets, Angelescu *et al* (1981b) have succeeded in analytically describing the physical solution which reduces the problem to the study of the stable manifold of the relevant hyperbolic fixed point.

For systems with competing interactions or structural transitions, the problem is much more difficult and analytic results are hard to come by (see, however, Aubry 1983).

When the self-consistency equations couple more than three layers each, the mathematical analysis gets much more involved, the associated map acting in a higher-dimensional space. It is not clear whether this brings about a corresponding complexity in the physical behaviour. It is likely that physically interesting orbits lie in a low-dimensional subspace and finding mechanisms through which this reduction is performed is important both physically and mathematically. This might be especially significant for truly short-range inhomogeneous systems in high spatial dimensions, where the local mean-field critical behaviour should set in.

A considerable mathematical simplification is obtained in the continuum approximation, which replaces the non-linear system by a one-dimensional Ginsburg-Landau equation. The latter corresponds to an integrable map and therefore its phase portrait is quite simple. Obviously the continuum theory loses information on the behaviour at the level of one layer, for example, it cannot describe the infinite sequence of distinct layer transitions which, however, do appear in the (non-integrable) discrete wetting theory.

Thus, even in the mean-field theory, the relationship between discrete and continuum theories is questionable. Intuitively, one expects the continuum approximation to be valid whenever the magnetisation variation is slow on the scale of lattice spacing, for instance deep into the bulk when the correlation length $\xi \gg 1$. Even if this is correct, it remains to be understood why one can thereby obtain reliable information on local critical behaviour. Establishing the range of validity of the continuum approximation within mean-field theory might be of relevance for finding its exact status in more sophisticated approaches. For Kac-Helfand models Angelescu *et al* (1981b) have found a mechanism by which the continuum mean-field theory can be recovered from the discrete one at the ordinary transition. This was related to the behaviour of the stable manifold of the 'ferromagnetic' fixed point near its bifurcation from the 'paramagnetic' one, which dictates the critical behaviour both on the scale of the correlation length and of the lattice constant. The problem of extending the range of applicability of this picture to more complicated models is intimately connected to the dimensionality reduction referred to above.

In this paper we answer these questions for the semi-infinite ferromagnetic Kac-Baker model (Kac and Helfand 1963, Baker 1963). The model assumes that the intralayer interactions are of mean-field type while spins in different layers interact via short-range (nearest-neighbour) forces. Every self-consistency equation of the model couples all layer magnetisations. As it looks the self-consistency system cannot be reduced to a recursion relation and thus be associated in an obvious way with a finite-dimensional dynamical system. We shall show, however, that in certain circumstances (near the ordinary transition line and around the special transition point) the behaviour of the solution is controlled by an area-preserving map (similar to the one related to the Kac-Helfand model and studied in Angelescu *et al* (1981b)). More specifically, in the critical region and in the neighbourhood of the interesting solution, the system behaves like a small perturbation to a main part which is a two-step recursion relation. As a consequence the mechanism by which the continuum approximation sets in is the same as in the case of a two-step recursion relation. This implies that, while the continuum approximation describes the phase diagram correctly, the magnetisation profile it predicts is not relevant on the surface and extraordinary transition lines but near their intersection point (special transition) it is.

2. Description of models and phase diagrams

The isotropic n-vector model with Kac–Baker interactions, which will be our concern here, consists of a set of classical n-dimensional 'spins', $\sigma_{i\alpha}$, of length $n^{1/2}$ ($1 \leq i \leq M$ labels the layers while $1 \leq \alpha \leq N$ labels the in-layer position of the spin) whose interaction energy is given by

$$\mathscr{H}_{M,N} = -\tfrac{1}{2} \sum J_{i\alpha,j\beta} \sigma_{i\alpha} \sigma_{j\beta} - \sum h_i \sigma_{i\alpha}^{(1)} \tag{2.1}$$

where

$$J_{i\alpha,j\beta} = (1/N)\delta_{ij}(1+\Delta\delta_{i1}) + \delta_{\alpha\beta}\delta_{|i-j|,1} \qquad \Delta > -1. \tag{2.2}$$

It can be seen that the spins in the same layer have mean-field type interactions of strength 1; we allowed the surface layer interaction strength to be $1+\Delta$ in order to make contact with other studies on surface critical behaviour (see, for instance, Binder 1983). Note also that the interaction among spins in different layers is short-ranged, namely of nearest-neighbour (NN) type. The magnetic field $h_i > 0$ acting on the spins in the ith layer and taken along the first axis in the spin space has only the usual transitory role and will eventually be set equal to zero (after performing the thermodynamic limit).

The limit $N \to \infty$ will provide an inhomogeneous mean-field model with M order parameters. Its free energy

$$F_M(\boldsymbol{h}) = -\lim_{N\to\infty} (\beta M N n)^{-1} \log \int \exp(-\beta \mathscr{H}_{M,N}) \prod_{i,\alpha} \mathrm{d}\sigma_{i\alpha}$$

is related (cf a general result on systems with mixed long-range and short-range interactions of Lebowitz and Penrose 1966) to the free energy $\tilde{F}_M(\boldsymbol{h})$ of the corresponding n-vector chain with NN interactions in an inhomogeneous external field through

$$F_M(\boldsymbol{h}) = \inf_{\boldsymbol{u}\in R^M} \left(\sum_{i=1}^{M} \tfrac{1}{2}(1+\Delta\delta_{i1})^{-1} u_i^2 + \tilde{F}_M(\boldsymbol{h}+\boldsymbol{u}) \right). \tag{2.3}$$

The semi-infinite model will thence be obtained by letting $M \to \infty$. We start by analysing the Ising model which, as will be seen, is in a certain sense special among the n-vector Kac–Baker models.

2.1. *The Ising model*

The extremum conditions in (2.3) involve the magnetisations $\mu_i(\boldsymbol{h}) = -\partial \tilde{F}_M(\boldsymbol{h})/\partial h_i$ of the Ising chain in an inhomogeneous field. These have been calculated by Costache (1976). One obtains

$$(1+\Delta\delta_{i1})^{-1} u_i = \mu_i(\boldsymbol{h}+\boldsymbol{u}) \equiv \tanh(x_i + y_{i-1}) \qquad (i = 1, \ldots, M) \tag{2.4}$$

where x_i, y_i are defined recursively via

$$x_{M+1} = 0 \qquad x_i = \beta(h_i + u_i) + f(x_{i+1}) \tag{2.5a}$$

$$y_0 = 0 \qquad f^{-1}(y_i) = \beta(h_i + u_i) + y_{i-1} \tag{2.5b}$$

with

$$f(x) = \tanh^{-1}(r \tanh x) \qquad r = \tanh \beta. \tag{2.6}$$

The layer magnetisations of the model are

$$m_i(\boldsymbol{h}) \equiv -\frac{\partial F_M}{\partial h_i}(\boldsymbol{h}) = \mu_i(\boldsymbol{h}+\boldsymbol{u}(\boldsymbol{h})) \tag{2.7}$$

where $\boldsymbol{u}(\boldsymbol{h})$ is the appropriate solution of (2.4) and thus $m_i(\boldsymbol{h})$ equals $u_i(\boldsymbol{h})$ for all but the first layer.

The main complication with (2.4) comes from the fact that the RHS couples all the unknowns. The peculiarity of the Ising model is related to the Ornstein-Zernicke property of the NN Ising chain: when considering the inverse problem of determining the external field $\boldsymbol{h}$ which produces prescribed magnetisations $\boldsymbol{m}$, i.e. when solving for $\boldsymbol{h}$ the system $m_i = \mu_i(\boldsymbol{h})$, $(i = 1, \ldots, M)$, it turns out that $h_i(\boldsymbol{m})$ depends in fact only on m_{i-1}, m_i and m_{i+1} (i.e. the direct correlation functions have the same range as the interaction). This fact has been observed by Percus (1977) who proved it in the thermodynamic limit and under an additional assumption. One can see, by looking at (2.5), that the property holds however unconditionally for every finite-length chain and every distribution of magnetisations of moduli less than 1. As a consequence, the self-consistency equations (2.4) are transformed into recursion relations, namely

$$\begin{aligned} &\beta[h_1 + (1+\Delta)m_1] = \tanh^{-1}\varphi(m_1, m_2) \\ &\beta(h_i + m_i) = \tanh^{-1}\varphi(m_i, m_{i+1}) - \tanh^{-1}\varphi(m_{i-1}, m_i) \qquad i = 2, \ldots, M \end{aligned} \tag{2.8}$$

where

$$\varphi(x, y) = 2(r^{-1}x - y)/\{r^{-1} - r + [(r^{-1} - r)^2 - 4(r^{-1} + r - 2)xy + 4(x-y)^2]^{1/2}\}. \tag{2.9}$$

The other n-vector chains are not exactly Ornstein-Zernicke systems. The study of their self-consistency equations, which is our main concern here, will be exemplified, for simplicity reasons, only on the spherical model (the $n \to \infty$ limit) where all the salient features of the analysis are preserved.

2.2. *The spherical model*

In order to get the spherical model for inhomogeneous systems one has to perform the spherical limit $n \to \infty$ of the n-vector models. Relying on the analysis of Knops (1973) and Angelescu *et al* (1979) the state of the system is determined once the 'spherical fields' are given. In our case the spherical fields in the thermodynamic limit $N \to \infty$, γ_i, can be obtained as the unique solution of the system

$$X_{ii}^{-1} = \beta(1 - m_i^2) \qquad i = 1, \ldots, M \tag{2.10}$$

where X is a tridiagonal matrix given by

$$X_{ij} = \gamma_i \delta_{ij} - \delta_{|i-j|,1} \tag{2.11}$$

subject to the condition

$$X - I_M - \Delta \cdot P > 0 \qquad P_{ij} = \delta_{ij}\delta_{i1} \qquad (I_M)_{ij} = \delta_{ij} \tag{2.12}$$

and m_i are the layer magnetisations defined in terms of γ_i by

$$(X - I_M - \Delta \cdot P)\boldsymbol{m} = \boldsymbol{h}. \tag{2.13}$$

One can alternatively view $\boldsymbol{m}$ as unknowns and determine $\boldsymbol{\gamma}$ in terms of $\boldsymbol{m}$ via (2.13), which substituted in (2.10) provides the self-consistency equations of the model.

Thus far we derived the equations for a slab; the semi-infinite system is hence obtained by letting $M \to \infty$. Taking into account that in both models the Griffiths inequalities hold (Griffiths 1967, Angelescu *et al* 1979), the limit $M \to \infty$ is straightforward: every layer magnetisation m_i converges as $M \to \infty$ monotonically and the limit magnetisations $\{m_i, i \geq 1\}$ will satisfy the infinite system of self-consistency equations, (2.8) and (2.10)-(2.13) respectively, where $M = \infty$. Likewise, the limit $\boldsymbol{h}\downarrow 0$ exists and satisfies these equations with $\boldsymbol{h} = 0$. However, in this process the strict inequality in (2.12) may no longer hold (see below).

The problem has been reduced to that of finding a solution of the infinite self-consistency system for $\boldsymbol{h} = 0$, subject to the conditions: $m_i \geq 0$ for all i and $\lim_{i\to\infty} m_i$ exists and equals m, the bulk spontaneous magnetisation at the given temperature. We take for granted that there exists only one solution with these properties. Though physically obvious, the uniqueness problem may present real mathematical difficulties if the equations are not recursion relations.

We conclude this section with a discussion of the phase diagrams of the two semi-infinite systems. To this aim, we shall find the domains in the parameter space (β, Δ) where the appropriate solution of the self-consistency equations is analytic of (β, Δ).

We start with the Ising model, where the Ornstein–Zernicke property allows a good qualitative discussion. Let us solve the second equation (2.8) for m_{i+1}: $m_{i+1} = F_\beta(m_{i-1}, m_i)$, and define the two-dimensional map T_β by

$$\begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{T_\beta} \begin{pmatrix} y \\ F_\beta(x, y) \end{pmatrix}. \tag{2.14}$$

Then, for every m_1, m_2 satisfying the first equation (2.8), one can obtain a solution by iterating T_β on $\binom{m_1}{m_2}$, i.e. a solution of (2.8) is given by an infinite trajectory of T_β. If we require moreover the existence of $\lim_{i\to\infty} m_i$, we have to look only for trajectories which converge to one of the fixed points of T_β. The latter are defined by the equation $x = F_\beta(x, x)$, i.e.

$$\beta x = \tanh^{-1} \varphi(x, x) - \tanh^{-1} r\varphi(x, x). \tag{2.15}$$

Equation (2.15) always has the solution $x = 0$, and this is the only solution if $\beta \leq \beta_c$ with β_c determined from

$$\beta_c \, e^{2\beta_c} = 1. \tag{2.16}$$

At β_c two other fixed points bifurcate from $\binom{0}{0}$, namely $\pm\binom{m(\beta)}{m(\beta)}$ with $m(\beta) > 0$ $(\beta > \beta_c)$. The free energy per site depends only on $\lim_{i\to\infty} m_i$ and is lower for $\pm m(\beta)$ than for 0. The symmetry breaking field $(\boldsymbol{h}\downarrow 0)$ throws away one solution, so we are left with one relevant fixed point, $\binom{m(\beta)}{m(\beta)}$. Thus, crossing β_c corresponds to the bulk transition and $m(\beta)$ is the bulk spontaneous magnetisation (compare with Kac and Helfand 1963).

To obtain the solution of (2.8) one still has to determine the starting point $\binom{m_1}{m_2}$ such that the trajectory be attracted by the relevant fixed point. By linearising T_β around the fixed points, one easily establishes that $\binom{0}{0}$ is hyperbolic for $\beta < \beta_c$ and $\binom{m(\beta)}{m(\beta)}$ is also hyperbolic for $\beta > \beta_c$, i.e. the corresponding tangent maps have one eigenvalue larger and one smaller than unity. The set of points attracted to a hyperbolic fixed point is the stable manifold of that point, in our case a certain curve $\mathcal{V}$ in the x, y plane whose tangent at the fixed point is given by the contracting direction of the tangent map. $\mathcal{V}$ is defined by a certain functional equation which is the main tool in deriving its various properties needed below. The physically interesting solution of

(2.8) appears thus as the trajectory T_β with starting point the intersection of the stable manifold $\mathscr{V}$ with the curve $\mathscr{C}_\Delta$ defined by the first equation (2.8). As $\mathscr{C}_\Delta$ has a simple dependence on Δ, while the fixed points and $\mathscr{V}$ depend only on β, the qualitative behaviour of the solution can be easily established as a function of (β, Δ).

For $\beta < \beta_c$, $\mathscr{C}_\Delta \cap \mathscr{V}$ contains the origin; if this is the only point in the intersection, which happens for Δ sufficiently small, the only solution is the trivial one, $m_i = 0$. However, if $\mathscr{C}_\Delta \cap \mathscr{V}$ also contains a point $\binom{m_1}{m_2}$ with $m_1, m_2 > 0$, the associated solution will have a smaller surface free energy and will thus be the thermodynamically stable solution; it will describe a phase with magnetised surface and layer magnetisations exponentially approaching zero (figure 1). This situation will appear for all $\beta < \beta_c$ if Δ is sufficiently large; the exact value of Δ, $\Delta_s(\beta)$, at which this solution bifurcates can be obtained by equating the slopes of $\mathscr{C}_\Delta$ and $\mathscr{V}$ at the origin. The result is

$$\Delta_s(\beta) = \frac{1-\beta}{2\beta} + \frac{1}{2}\left[\left(\frac{1-\beta}{\beta}\right)^2 - \frac{4r}{\beta(r^{-1}-r)}\right]^{1/2} \qquad \beta < \beta_c. \tag{2.17}$$

As expected $\Delta_s(\beta) \sim \beta^{-1}$ for $\beta \downarrow 0$ (the surface decouples completely). In the continuum approximation this cannot be true, contrary to the assertion made by Lubensky and Rubin (1975).

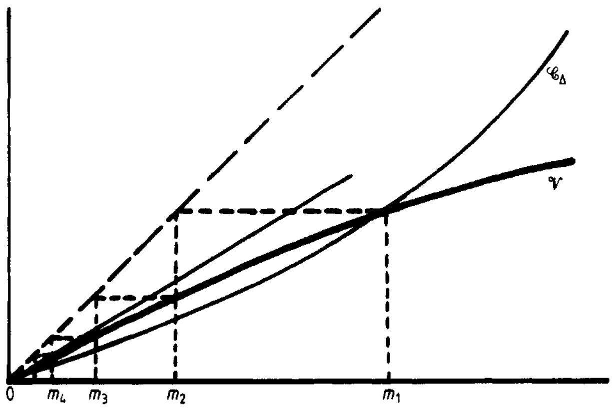

Figure 1. The stable manifold and the orbit corresponding to the surface-magnetised phase.

For $\beta > \beta_c$ and Δ sufficiently small (large), one has $m_1 < m(\beta)$ ($m_1 > m(\beta)$) so the solution increasingly (decreasingly) approaches $m(\beta)$. In all cases one has a magnetised bulk and the solution depends analytically on Δ (because $\mathscr{C}_\Delta$ itself does) (figure 2).

The resulting phase diagram is depicted in figure 3. Phase I is paramagnetic, III ferromagnetic, II has surface but not bulk magnetisation. The various transitions are: I-II ($\beta < \beta_c$, $\Delta = \Delta_s(\beta)$) surface transition; I-III ($\beta = \beta_c$, $\Delta < \Delta_c = \Delta_s(\beta_c)$) ordinary transition; II-III ($\beta = \beta_c$, $\Delta > \Delta_c$) extraordinary transition. (β_c, Δ_c) is called the special transition point.

In the case of the spherical model (and for all n-vector models with $n \geqslant 2$) another approach is needed. We start again from the remark that (2.10) and (2.13) (where $\boldsymbol{h} = 0$) always have the solution $\boldsymbol{m} = 0$; the corresponding matrix X_0 should satisfy

$$(X_0^{-1})_{ii} = \beta \qquad i = 1, 2, \ldots \tag{2.18}$$

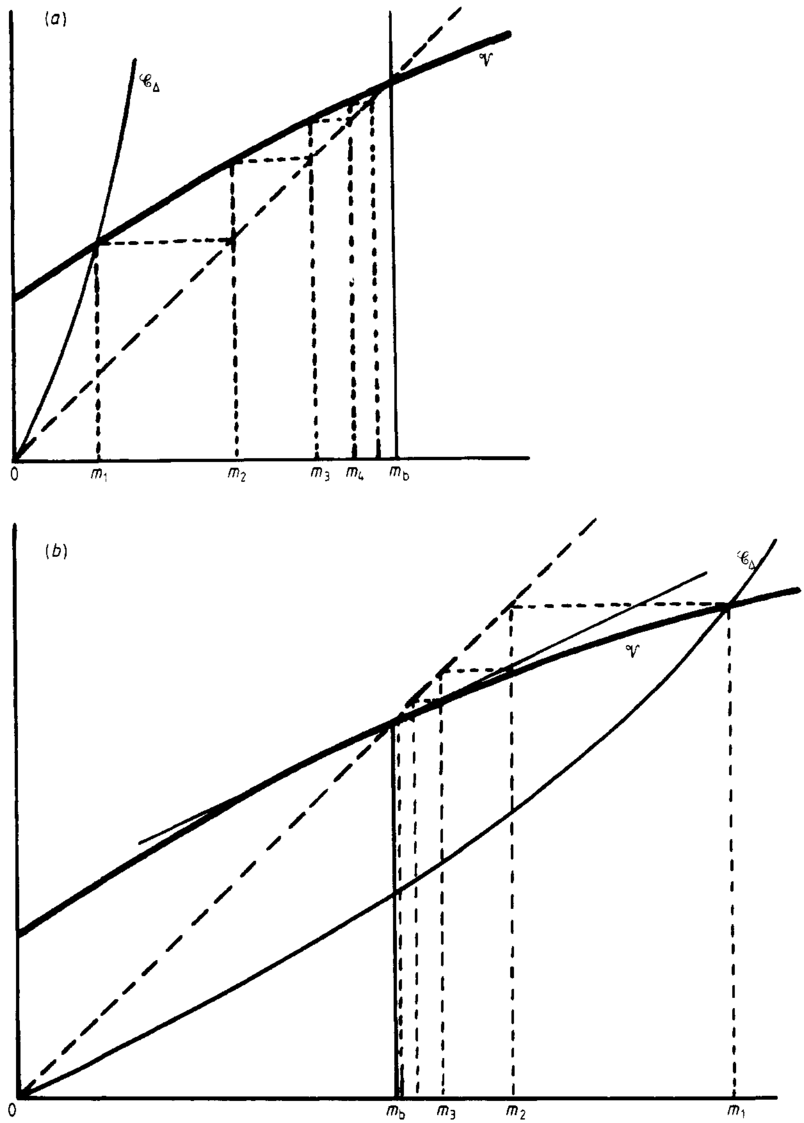

Figure 2. The stable manifolds and the orbits corresponding to the ferromagnetic phase when (a) $m_1 < m$, (b) $m_1 > m$.

which is solved explicitly

$$\gamma_1^0 = \kappa_0^{-1} \qquad \gamma_i^0 = \kappa_0 + \kappa_0^{-1} \qquad i \geq 2 \tag{2.19}$$

where $0 < \kappa_0 = \kappa_0(\beta) < 1$ is the solution of the equation

$$(\kappa^{-1} - \kappa)^{-1} = \beta$$

i.e.

$$\kappa_0(\beta) = -\frac{1}{2\beta} + (1 + 1/4\beta^2)^{1/2}. \tag{2.20}$$

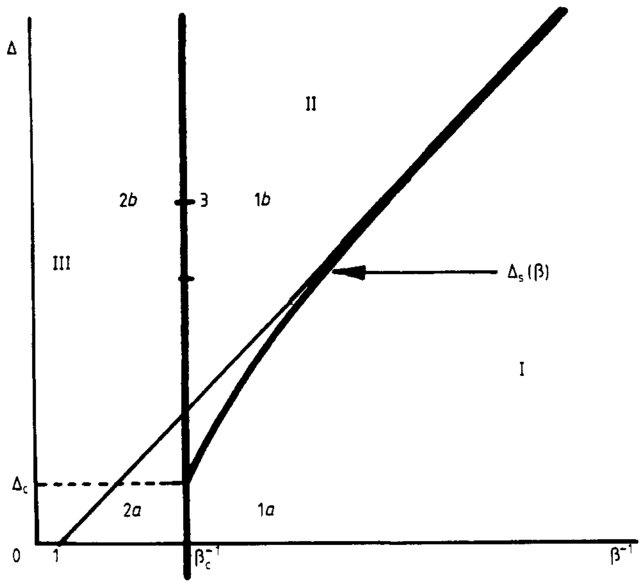

Figure 3. The phase diagram.

Indeed, the matrix X_0 with diagonal entries (2.19) is positive definite, its continuous spectrum is the segment $\mathscr{S}_0 = [\kappa_0 + \kappa_0^{-1} - 2, \kappa_0 + \kappa_0^{-1} + 2]$ and its inverse is given by

$$(X_0^{-1})_{ij} = (\kappa_0^{-1} - \kappa_0)^{-1} \cdot \kappa_0^{|i-j|} \qquad i, j = 1, 2, \ldots . \tag{2.21}$$

However, this is not the solution of the self-consistency equations unless $X_0 - I - \Delta \cdot P \geqslant 0$. The spectrum of the latter consists of a continuous component $\mathscr{S}_0 - 1 = [\kappa_0 + \kappa_0^{-1} - 3, \kappa_0 + \kappa_0^{-1} + 1]$ and, if $(X_0)_{22} - (X_0)_{11} = \kappa_0 + \Delta > 1$, of an isolated eigenvalue $\lambda_0 = \kappa_0 + \kappa_0^{-1} - 1 + (\kappa_0 + \Delta) - (\kappa_0 + \Delta)^{-1}$. Thus we must require $\kappa_0 + \kappa_0^{-1} - 3 \geqslant 0$ for $\kappa_0 + \Delta \leqslant 1$ and $\lambda_0 \geqslant 0$ for $\kappa_0 + \Delta > 1$. This corresponds to region I in figure 3 whose boundary is given by

$$\{\beta = \beta_c = 5^{-1/2}, \Delta \leqslant \Delta_c = \tfrac{1}{2}(5^{1/2} - 1)\} \cup \left\{ \beta = \beta_s(\Delta) = \left[\Delta + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{\Delta(1+\Delta)}\right)^{1/2}\right]^{-1}, \Delta > \Delta_c \right\}. \tag{2.22}$$

In region II $\lambda_0 < 0$ but the continuous spectrum $\mathscr{S}_0 - 1$ is strictly above zero; the matrix X corresponding to the solution is obtained by perturbing X_0 with a 'potential' $V = \Gamma - \Gamma_0$ which vanishes when $i \to \infty$, such that the isolated eigenvalue is shifted up to zero (thereby the continuous spectrum is not affected). The corresponding eigenvector of $X - I - \Delta \cdot P$ is $\boldsymbol{m}$. Thus, $\lim_{i \to \infty} m_i = 0$ and $m_i > 0$ for all i (because X has non-positive off-diagonal entries) which corresponds to a phase with only surface magnetisation. Finally, in region III, 0 is inside $\mathscr{S}_0 - 1$ and no potential vanishing at infinity can make the matrix positive definite. In this case $X = X_0(\beta_c) - V$ with V vanishing at infinity (cf § 3). In particular, $\lim_{i \to \infty} (X^{-1})_{ii} = \beta_c$, which means that $m = \lim_{i \to \infty} m_i$ exists and satisfies $\beta_c = \beta(1 - m^2)$, i.e. m is the bulk magnetisation of the spherical model.

3. The critical behaviour at the ordinary transition

In this section we shall study the critical behaviour of the layer magnetisations at the ordinary transition. We shall show that, as expected, the continuum approximation is good there. The study of the Kac-Baker model with $n = 1$, (2.8), is similar to that of the Kac-Helfand model done by Angelescu *et al* (1981b), up to inessential technical complications related to the different form of $F_\beta(x, y)$ in (2.14). Its critical behaviour is dictated by the behaviour of the stable manifold for $\beta \downarrow \beta_c$. We shall therefore consider in some detail only the spherical model. In this case also, we isolate a finite-dimensional map which controls the transition, i.e. whose trajectories provide good approximations of the exact solution in the critical region. This picture might hopefully hold in a wider context, e.g. for short-range models in high spatial dimension.

The equations of the semi-infinite model near the ordinary transition are

$$\operatorname{diag}(X^{-1}) = \beta(\mathbf{1} - \boldsymbol{mm}) \qquad X\boldsymbol{m} = \boldsymbol{m} \qquad \inf\operatorname{spec}(X - I) = 0 \tag{3.1}$$

where we have put for simplicity $\Delta = 0$. Here, $\operatorname{diag} M$ denotes the vector whose components are the diagonal entries of the matrix M, $\boldsymbol{xy}$ is the vector with components $x_i y_i$, and $\mathbf{1}_i = 1$. If $X_c \equiv X_0(\beta_c)$, then $\operatorname{diag}(X_c^{-1}) = \beta_c \mathbf{1} = \beta(1 - m^2)\mathbf{1}$ and

$$(X_c)_{11} = \tfrac{1}{2}(3 + \sqrt{5}) \qquad (X_c)_{ii} = 3 \qquad i \geqslant 2. \tag{3.2}$$

Now, denoting

$$V = X_c - X \qquad \boldsymbol{v} = \operatorname{diag} V \qquad \boldsymbol{x} = \boldsymbol{m}/m \tag{3.3}$$

and using the perturbation formula

$$(X_c - V)^{-1} - X_c^{-1} = X_c^{-1} V X_c^{-1} + X_c^{-1} V X_c^{-1} V (X_c - V)^{-1} \tag{3.4}$$

we transform (3.1) into

$$A^{-1}\boldsymbol{v} - \beta m^2(\mathbf{1} - \boldsymbol{xx}) = -\operatorname{diag}[X_c^{-1} V X_c^{-1} V (X_c - V)^{-1}] \tag{3.5a}$$

$$v_i = [(X_c - I)\boldsymbol{x}]_i / x_i \qquad x_i > 0 \qquad i = 1, 2, \ldots. \tag{3.5b}$$

The matrix A^{-1} in (3.5a) is the Schur square of X_c^{-1}, i.e. $(A^{-1})_{ij} = (X_c^{-1})_{ij}^2$; it is therefore positive definite. Its inverse is tridiagonal, which expresses an approximate Ornstein-Zernicke property of the model (A_{ij} is the lowest order in the expansion of the direct correlation function around $\boldsymbol{m} = 0$) and is given by

$$A = \tfrac{1}{3}\sqrt{5}[(X_c - I) + 5D] \qquad D = I + \tfrac{1}{5}(\sqrt{5} - 2)P. \tag{3.6}$$

It is precisely because A has finite range which enables one to approximately reduce the system to a dynamical system, implying mean-field critical behaviour (one would expect the same behaviour of the NN spherical model in spatial dimension $d \geqslant 5$, where the corresponding A has exponential decay). Applying A to (3.5a) and substituting (3.5b), our system is brought to the form

$$\boldsymbol{Q}_t(\boldsymbol{x}) = t^4 \boldsymbol{x} \boldsymbol{R}_t(\boldsymbol{x}) \tag{3.7}$$

where we have redefined for convenience the small parameter

$$t = (\beta\sqrt{5}/3)^{1/2} m \tag{3.8}$$

and

$$\boldsymbol{Q}_t(\boldsymbol{x}) = (X_c - I)\boldsymbol{x} - 5t^2 D\boldsymbol{x}(\mathbf{1} - \boldsymbol{xx}) - \tfrac{1}{2}(\sqrt{5} - 1)t^2 P\boldsymbol{x} \tag{3.9}$$

$$\boldsymbol{R}_t(\boldsymbol{x}) = -t^{-2}(X_c - I)(\boldsymbol{xx}) - A \operatorname{diag}\left(X_c^{-1} \frac{V}{t^2} X_c^{-1} \frac{V}{t^2} (X_c - V)^{-1} \right). \tag{3.10}$$

Let us stress that in this way we isolated on the LHS $\boldsymbol{Q}_t(\boldsymbol{x})_i$ which depends only on x_{i-1}, x_i, x_{i+1}. The task which we shall endeavour next is to show that the RHS is a small perturbation thereof. The core of the argument is that $\boldsymbol{R}_t(\boldsymbol{x}^0(t))$ remains bounded when $t\downarrow 0$ if $\boldsymbol{x}^0(t)$ is the positive solution of the equation

$$\boldsymbol{Q}_t(\boldsymbol{x})=\boldsymbol{0}. \tag{3.11}$$

This requires a close study of the properties of $\boldsymbol{x}^0(t)$. But, as explained in § 2, the solutions of (3.11) are related to the trajectories of the map $T_t: R^2 \to R^2$ given by

$$T_t\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}y\\2y-x-5t^2y(1-y^2)\end{pmatrix} \qquad t>0. \tag{3.12}$$

T_t is invertible and area preserving. Its fixed points are $\binom{0}{0}$ and $\pm\binom{1}{1}$. The tangent map of T_t at $\boldsymbol{e}=\binom{1}{1}$

$$T_t'(\boldsymbol{e})=\begin{pmatrix}0 & 1\\ -1 & 2+10t^2\end{pmatrix} \tag{3.13}$$

has eigenvalues $\lambda_t<1$ and $\lambda_t^{-1}>1$ and eigenvectors $\binom{1}{\lambda_t}$ and $\binom{\lambda_t}{1}$, respectively. We note that

$$\lambda_t=1-\sqrt{10}t+O(t^2). \tag{3.14}$$

Thus, $\boldsymbol{e}$ is a hyperbolic fixed point of T_t, which becomes degenerate when $t\downarrow 0$. The following lemma summarises the information we need about its stable manifold.

Lemma 1. The connected component of the stable manifold of $\boldsymbol{e}$ contained in the positive quadrant of R^2 and containing $\boldsymbol{e}$ is the graph of a C^∞-function $\varphi_t: R_+\to R_+$. φ_t is strictly increasing, contracting ($\varphi_t'(x)<1$) and concave. Moreover, uniformly on compacts

$$\lim_{t\downarrow 0} t^{-1}[\varphi_t(x)-x]=h(x)\equiv(\tfrac{5}{2})^{1/2}(1-x^2). \tag{3.15}$$

The proof of the lemma can be adapted from Angelescu *et al* (1981b). For the reader's convenience, we outline it in appendix 1. As an immediate consequence, we have the following information about $\boldsymbol{x}^0(t)$.

Corollary 1. The equation $\boldsymbol{Q}_t(\boldsymbol{x})=\boldsymbol{0}$ has a unique positive solution, $\boldsymbol{x}^0(t)$, such that $\lim_{i\to\infty} x^0(t)_i=1$. For $t\downarrow 0$ the following limits exist:

$$\lim_{t\downarrow 0} t^{-1}x^0(t)_i=\left(\frac{5}{2}\right)^{1/2}\frac{\sqrt{5}+1}{2}+i-1 \qquad i=1,2,\ldots \tag{3.16}$$

$$\lim_{t\downarrow 0, i\to\infty, it\to z} x^0(t)_i=\mu(z) \qquad \text{(uniformly on compacts in } z\geqslant 0). \tag{3.17}$$

Here the scaled magnetisation profile, $\mu(z)$, is the solution of the differential equation

$$\mathrm{d}\mu/\mathrm{d}z=h(\mu) \qquad \mu(0)=0 \tag{3.18}$$

i.e. $\mu(z)=\tanh(\frac{5}{2})^{1/2}z$.

Proof. The trajectory attracted by $\boldsymbol{e}$ and completely contained in the positive quadrant should start on the graph of φ_t. We have therefore to intersect the latter with $\boldsymbol{Q}_t(\boldsymbol{x})_1=0$, i.e. $x^0(t)_1$ is the (unique) positive solution of the equation

$$\tfrac{1}{2}(1+\sqrt{5})x_1-5t^2x_1(1-x_1^2)D_{11}-t^2\tfrac{1}{2}(\sqrt{5}-1)x_1=\varphi_t(x_1). \tag{3.19}$$

From (3.19) and (3.15), we have

$$x^0(t)_1 \sim t\left(\frac{5}{2}\right)^{1/2}\frac{\sqrt{5}+1}{2}$$

for $t\downarrow 0$. Also, $x^0(t)_i = \varphi_t^{\circ(i-1)}(x^0(t)_1) \sim x^0(t)_1 + (i-1)th(0)$, which gives (3.16). The limit (3.17) equals $\lim_{i\to\infty} \varphi_{z/i}^{\circ i}(x^0(t)_1) = \lim_{i\to\infty} \varphi_{z/i}^{\circ i}(0)$; the non-linear Trotter-Chernoff formula (Brezis and Pazy 1970) applied to the family φ_t of contradictions of $[0, \infty)$, identifies the latter with the 'evolution' of 0 at 'time' z under the semigroup generated by $\lim_{t\downarrow 0} t^{-1}(\varphi_t - \varphi_0) = h$, i.e. with the solution of the Cauchy problem (3.18) evaluated at z.

Later on we shall need further properties of $\boldsymbol{x}^0(t)$, among which the exponential approach of $x^0(t)_i$ to 1 as $i\to\infty$ plays a prominent role. In order to control the latter, we define now a suitable family of Banach spaces indexed by t.

Let $0<\zeta<1$ be an arbitrary, but fixed, number and

$$p_t = \frac{1-\varphi_t(\zeta)}{1-\zeta}. \tag{3.20}$$

We define B_t as the Banach space of all sequences $\boldsymbol{\xi} = \{\xi_i;\ i=1,2,\ldots\}$ with

$$\|\boldsymbol{\xi}\|_t \equiv \sup_{i=1,2,\ldots} p_t^{-i}|\xi_i| < \infty.$$

The adequacy of this definition is shown by the following estimate.

Corollary 2. For some t_0, $C>0$ (depending only on ζ), we have for all $t<t_0$

$$\mathbf{1} - \boldsymbol{x}^0(t) \in B_t \qquad \text{and} \qquad \|\mathbf{1} - \boldsymbol{x}^0(t)\|_t \leq C$$

where $\mathbf{1}$ denotes the sequence with all terms equal to unity.

Proof. Clearly

$$\lambda_t < p_t < 1 \qquad \text{and} \qquad \lim_{t\downarrow 0} \frac{1-p_t}{t} = \frac{h(\zeta)}{1-\zeta} > 0. \tag{3.21}$$

Because φ_t is concave, we have $\varphi_t(z) > 1 - p_t(1-z)$ for $z>\zeta$ and $\varphi_t(z) > (1-p_t)(1-\zeta)+z$ for $z<\zeta$. Hence, remembering that x_i^0 are obtained by iterating φ_t on x_1^0 and defining i_t as the greatest i for which $x_i^0 < \zeta$, we have

$$x_i^0 > i(1-p_t)(1-\zeta) \qquad \text{for} \qquad i \leq i_t$$

$$x_i^0 > 1 - p_t^{i-i_t}(1-\zeta) \qquad \text{for} \qquad i > i_t.$$

Hence

$$\|\mathbf{1} - \boldsymbol{x}^0(t)\|_t \leq \max\{p_t^{-i_t}(1-\zeta);\ p_t^{-i}[1 - i(1-p_t)(1-\zeta)],\ i\leq i_t\} \leq p_t^{-i_t}(1-\zeta).$$

Now, $i_t < \zeta/(1-p_t)(1-\zeta)$, which together with the properties of p_t given above, shows that $p_t^{-i_t} \sim (1-at)^{-bt^{-1}} \to \mathrm{e}^{ab}$ for $t\to 0$ for some constants $a, b>0$, which proves corollary 2.

We are now ready to prove that $\boldsymbol{R}_t(\boldsymbol{x}^0(t))$ is bounded for $t\downarrow 0$.

Corollary 3. There exist t_0, $C>0$ such that, for $t<t_0$:

$$\|\boldsymbol{R}_t(\boldsymbol{x}^0(t))\|_t < C. \tag{3.22}$$

Proof. From (3.11) we have

$$t^{-2}\boldsymbol{v}(\boldsymbol{x}^0)=5D(\mathbf{1}+\boldsymbol{x}^0)(\mathbf{1}-\boldsymbol{x}^0)+\tfrac{1}{2}(\sqrt{5}-1)P\mathbf{1}.$$

Therefore corollary 2 and $x_i^0<1$ imply $t^{-2}\boldsymbol{v}(\boldsymbol{x}^0)\in B_t$ for $t<t_0$, and

$$\|t^{-2}\boldsymbol{v}(\boldsymbol{x}^0)\|_t\leqslant C \qquad t<t_0 \tag{3.23}$$

for some $C>0$. Using the boundedness of A and X_c^{-1} as operators in B_t, this allows us to bound uniformly the t-norm of the second term in (3.10). For the first term of $\boldsymbol{R}_t$, we use the identity

$$t^{-2}[(X_c-I)(\boldsymbol{x}\boldsymbol{x})]_i=2x_i^2t^{-2}v_i(\boldsymbol{x})-t^{-2}(\nabla\boldsymbol{x})_{i-1}^2-t^{-2}(\nabla\boldsymbol{x})_i^2 \qquad i\geqslant 2 \tag{3.24}$$

where $(\nabla\boldsymbol{x})_i=x_{i+1}-x_i$. Now

$$t^{-1}(\nabla\boldsymbol{x}^0)_i=t^{-1}[\varphi_t(x_i^0)-x_i^0]\leqslant t^{-1}(1-\lambda_t)(1-x_i^0)$$

where we used the concavity of φ_t and $\varphi_t'(1)=\lambda_t$, which is estimated by corollary 2. Hence $t^{-1}\nabla\boldsymbol{x}^0\in B_t$ and

$$\|t^{-1}\nabla\boldsymbol{x}^0\|_t\leqslant C \qquad t<t_0. \tag{3.23'}$$

Moreover, $[(X_c-I)(\boldsymbol{x}\boldsymbol{x})]_1$ is bounded by a constant times $(\boldsymbol{x}^0\boldsymbol{x}^0)_1$, which is $O(t^2)$ by (3.16).

The remainder of this section is devoted to solving (3.7) for small t. This will be done iteratively, starting with $\boldsymbol{x}^0(t)$ and solving at level k the equation

$$\boldsymbol{Q}_t(\boldsymbol{x})=t^4\boldsymbol{x}\boldsymbol{R}_t(\boldsymbol{x}^{k-1}(t)) \tag{3.25}$$

whose solution $\boldsymbol{x}^k(t)$ will be shown to converge for $k\to\infty$. Thus we must consider equations of the form

$$\boldsymbol{Q}_t(\boldsymbol{x})=t^4\boldsymbol{x}\boldsymbol{r}. \tag{3.26}$$

We shall show that its solution has the same critical behaviour as $\boldsymbol{x}^0(t)$ provided $\boldsymbol{r}=\boldsymbol{r}(t)\in B_t$ and $\|\boldsymbol{r}\|_t$ is bounded for $t\downarrow 0$.

To this aim, we linearise $\boldsymbol{Q}_t(\boldsymbol{x}^0+\boldsymbol{\xi})$ around $\boldsymbol{\xi}=\mathbf{0}$

$$\boldsymbol{Q}_t(\boldsymbol{x}^0+\boldsymbol{\xi})=Q_t'(\boldsymbol{x}^0)\boldsymbol{\xi}+t^2\boldsymbol{G}(\boldsymbol{\xi})$$

where $\boldsymbol{G}(\boldsymbol{\xi})$ is quadratic in $\boldsymbol{\xi}$ around $\boldsymbol{\xi}=\mathbf{0}$. It is shown in appendix 2 that $Q_t'(\boldsymbol{x}^0)$ is invertible as an operator in B_t and the norm of $t^2Q_t'(\boldsymbol{x}^0)^{-1}$ is uniformly bounded for t less than some t_0. It follows that $Q_t'(\boldsymbol{x}^0)-t^4\hat{r}$ (where $\hat{r}$ is the diagonal matrix $\delta_{ij}r_i$) is also invertible and $\|t^2[Q_t'(\boldsymbol{x}^0)-t^4\hat{r}]^{-1}\|_t$ is less than some constant C_1 for $t\leqslant t_0$, provided $t^2\|\hat{r}\|_t$ is sufficiently small. Hence (3.26) can be put in the form

$$\boldsymbol{\xi}=t^2[Q_t'(\boldsymbol{x}^0)-t^4\hat{r}]^{-1}(t^2\boldsymbol{x}^0\boldsymbol{r}-\boldsymbol{G}(\boldsymbol{\xi}))\equiv\boldsymbol{H}_t^r(\boldsymbol{\xi}) \tag{3.26'}$$

which is solved by the contraction principle (Banach's fixed point theorem). Indeed, because $\boldsymbol{G}(\boldsymbol{\xi})$ is quadratic around $\boldsymbol{\xi}=\mathbf{0}$, $\boldsymbol{H}_t^r$ leaves invariant a certain ball $\|\boldsymbol{\xi}\|_t<\varepsilon$ and its Lipschitz constant on that ball is less than, say, $\frac{1}{2}$. In this way one obtains lemma 2.

Lemma 2. There exist positive constants ε, δ, t_0, C such that, for all $t<t_0$ and all $\boldsymbol{r}\in B_t$ with $t^2\|\boldsymbol{r}\|_t<\delta$, (3.26) has a unique solution $\boldsymbol{x}^r(t)=\boldsymbol{x}^0(t)+\boldsymbol{\xi}^r(t)$ with the property $\|\boldsymbol{\xi}^r(t)\|_t<\varepsilon$. Moreover, for any such $\boldsymbol{r}_1$, $\boldsymbol{r}_2$:

$$\|\boldsymbol{\xi}^{r_1}(t)-\boldsymbol{\xi}^{r_2}(t)\|_t\leqslant Ct^2\|\boldsymbol{r}_1-\boldsymbol{r}_2\|_t. \tag{3.27}$$

In particular,

$$\|\boldsymbol{\xi}^r(t)\|_t\leqslant Ct^2\|\boldsymbol{r}\|_t. \tag{3.27'}$$

The estimate (3.27) follows from

$$\begin{aligned}\|\xi^{r_1}-\xi^{r_2}\|_t &= \|H_t^{r_1}(\xi^{r_1})-H_t^{r_1}(\xi^{r_2})+H_t^{r_1}(\xi^{r_2})-H_t^{r_2}(\xi^{r_2})\|_t \\ &\leq \tfrac{1}{2}\|\xi^{r_1}-\xi^{r_2}\|_t+\|t^2[Q_t'(x^0)-t^4\hat{r}_1]^{-1}-t^2[Q_t'(x^0) \\ &\quad -t^4\hat{r}_2]^{-1}\|_t\|t^2x^0r_1-G(\xi^{r_2})\|_t+C_1t^2\|\hat{x}^0(r_1-r_2)\|_t \\ &\leq \tfrac{1}{2}\|\xi^{r_1}-\xi^{r_2}\|_t+\tfrac{1}{2}Ct^2\|r_1-r_2\|_t\end{aligned}$$

where we used for the middle term the second resolvent formula.

We can now control the iterative process for solving (3.7). When performing one iterative step, (3.25), one goes from an equation like (3.26) to another one with r replaced by $r'=F_t(r)$, where

$$F_t(r)=R_t(x^r(t)). \tag{3.28}$$

Lemma 3. For every $\rho>0$, there exists $t_1>0$ such that F_t is a contraction of the ball $\|r\|_t<\rho/t$ for all $t<t_1$.

To prove this, let ε and δ be chosen as in lemma 2. That F_t leaves the ball invariant can be seen by estimating the norm of $R_t(x^r)$ in the same way as we did for $R_t(x^0)$. Indeed, the estimates (3.23) and (3.23′) hold true for x^r with possibly ρ-dependent constants as a consequence of (3.27′). For instance, as x^r satisfies (3.26),

$$t^{-2}v(x^r)=t^{-2}v(x^0)+t^2r+5D(x^0x^0-x^rx^r)$$

in which the last term has norm less than $5D_{11}(2+\|\xi^r\|_t)\|\xi^r\|_t$. Likewise

$$t^{-1}\|\nabla x^r\|_t\leq t^{-1}\|\nabla x^0\|_t+2t^{-1}\|\xi^r\|_t\leq C(1+2\rho).$$

Thus

$$t\|F_t(x^r)\|_t\leq C_1(\rho)t \tag{3.29}$$

and this is less than ρ if $t<t_1\leq\rho/C_1(\rho)$.

The Lipschitz constant of F_t is estimated analogously using (3.27)

$$L(F_t)\leq C_2(\rho)t$$

and this can be made arbitrarily small by choosing t_1 sufficiently small.

As a consequence of lemma 3 the sequence $r^0=\mathbf{0}$, $r^k=F_t(r^{k-1})$ converges in B_t for all $t<t_1$. Let $r(t)=\lim_{k\to\infty}r^k(t)$. From the contraction principle

$$\|r(t)\|_t\leq(1-L(F_t))^{-1}\|F_t(\mathbf{0})\|_t$$

which is bounded uniformly in t by (3.22). According to lemma 2, ξ^{r^k} converges to $\xi(t)\in B_t$ and $x(t)=x^0(t)+\xi(t)$ is a solution of (3.7). Moreover, inequality (3.27′) implies

$$\|x(t)-x^0(t)\|_t\leq Ct^2. \tag{3.30}$$

In conclusion, we have constructed the solution of (3.7) with the required properties. One can see easily that (3.30) ensures that corollary 1 holds true for $x(t)$, i.e. the exact solution has the critical behaviour dictated by the associated two-dimensional map (3.12).

4. The critical scaling at the special transition

When (β, Δ) approaches (β_c, Δ_c), there are various scaling regimes depending on the path we follow in the parameter space. All these regimes can be read off the dynamical system associated with (2.11)-(2.13), which again controls the behaviour of the solution, as was the case at the ordinary transition considered in the previous section. We restrain from giving formal proofs of this assertion: they can be modelled on those in § 3. We mention however that the argument was founded on two important facts: (i) $\boldsymbol{R}_t(\boldsymbol{x}^0(t))$ has bounded $\|\cdot\|_t$-norm for $t\downarrow 0$; (ii) $t^2 Q_t'(\boldsymbol{x}^0(t))^{-1}$ as an operator in B_t has a norm bounded uniformly in t for $t\downarrow 0$. In the cases considered below, (i) follows as before from the structure of the 'unperturbed' equation, $\boldsymbol{Q}_t(\boldsymbol{x})=\boldsymbol{0}$, with a slightly modified definition of $\boldsymbol{Q}_t$, as shortly explained in the text. The essentials of the calculations leading to (ii) are given in appendix 2.

4.1. Approaching (β_c, Δ_c) from the ferromagnetic phase

For $\beta > \beta_c$ and with the same definition of the small parameter t, (3.8), we assume that

$$\Delta - \Delta_c = \delta t \qquad \text{as } t\downarrow 0 \tag{4.1}$$

for some fixed $\delta \in R$. Following the same procedure as for deriving (3.7) from (3.1), we obtain again an equation like (3.7), where, however, we choose

$$\begin{aligned}\boldsymbol{Q}_t(\boldsymbol{x}) = {} & (X_c - I - \Delta_c P)\boldsymbol{x} - t\delta P\boldsymbol{x} - 5t^2 D\boldsymbol{x}(\boldsymbol{1}-\boldsymbol{x}\boldsymbol{x}) \\ & - t^2\tfrac{1}{2}(\sqrt{5}-1)P\boldsymbol{x} + t^2\boldsymbol{x}P(X_c - I)(\boldsymbol{x}\boldsymbol{x})\end{aligned} \tag{4.2}$$

while $\boldsymbol{R}_t(\boldsymbol{x})$ is obtained by adding $t^2\boldsymbol{x}P(X_c-I)(\boldsymbol{x}\boldsymbol{x})$ to (3.10), with

$$v_i = [(X_c - I - (\Delta_c + \delta t)\cdot P)\boldsymbol{x}]_i / x_i . \tag{4.3}$$

When solving the equation $\boldsymbol{Q}_t(\boldsymbol{x}) = \boldsymbol{0}$, we have to find a trajectory of the same map (3.12) in the stable manifold of $\boldsymbol{e}$, but now the equation for $x^0(t)_1$ is changed into

$$x_1(1-\delta t) + \mathrm{O}(t^2 x_1) = \varphi_t(x_1). \tag{4.4}$$

By lemma 1, this implies that $x^0(t)_1$ converges to the solution $\tilde{x}_1(\delta)$ of the equation $\delta x + h(x) = 0$ and, in fact, that all $x^0(t)_i$ with fixed i have the same limit

$$\lim_{t\downarrow 0} x^0(t)_i = \tilde{x}_1(\delta) = \frac{1}{\sqrt{10}}[\delta + (\delta^2+10)^{1/2}] \qquad i = 1, 2, \ldots . \tag{4.5}$$

Applying as before the Trotter-Chernoff formula to find the asymptotic magnetisation profile, we find

$$\lim_{t\downarrow 0,\, i\to\infty,\, it\to z} x^0(t)_i = \mu(z, \delta) \tag{4.6}$$

where $\mu(\cdot, \delta)$ is the solution of the Cauchy problem

$$\mathrm{d}\mu/\mathrm{d}z = h(\mu) \qquad \mu(0) = \tilde{x}_1(\delta). \tag{4.7}$$

Explicitly

$$\mu(z, \delta) = \begin{cases} \tanh(\sqrt{10}/2)(z + z_0(\delta)) & \text{for } \delta < 0,\ z_0(\delta) = (2/\sqrt{10})\tanh^{-1}\tilde{x}_1(\delta) \\ 1 & \text{for } \delta = 0 \\ \operatorname{cotanh}(\sqrt{10}/2)(z + z_0(\delta)) & \text{for } \delta > 0,\ z_0(\delta) = (2/\sqrt{10})\operatorname{cotanh}^{-1}\tilde{x}_1(\delta). \end{cases} \tag{4.8}$$

Equations (4.5) and (4.6) hold for the solution of the complete equation (3.7) and summarise the information on the critical behaviour. We mention that, due to the fact that $x^0(t)_1$ approaches a non-zero limit, we were forced to include the last term of (4.2) into the definition of $\boldsymbol{Q}_t$; what is left in $\boldsymbol{R}_t$ has the right order of magnitude, by the same argument as before (because the identity (3.24) is true for $i \geq 2$), while affecting the first component of $\boldsymbol{Q}_t$ by a $O(t^2)$ term does not change the behaviour of $\boldsymbol{x}^0(t)$.

4.2. Approaching (β_c, Δ_c) from the surface-magnetised phase

For $\beta < \beta_c$, it is convenient to look at X as a perturbation around the solution $X_0(\beta)$ of (2.18) (determined in (2.19) and (2.20)), $X = X_0(\beta) - V$, with V diagonal and vanishing for $i \to \infty$. Using the perturbation formula (3.4) and defining $[A(\beta)^{-1}]_{ij} = [X_0(\beta)^{-1}]_{ij}^2$, (2.10) and (2.13) become

$$V + \beta A(\beta)(\boldsymbol{mm}) = -A(\beta)\operatorname{diag}[X_0(\beta)^{-1} V X_0(\beta)^{-1} V (X_0(\beta) - V)^{-1}] \tag{4.9}$$

$$v_i = [(X_0(\beta) - I - \Delta \cdot P)\boldsymbol{m}]_i / m_i. \tag{4.10}$$

We have

$$X_0(\beta) = X_c + \tfrac{5}{3}\sqrt{5}(\beta_c - \beta) + O[(\beta_c - \beta)^2] \tag{4.11}$$

where the correction is diagonal and with equal entries for $i \geq 2$, and

$$A(\beta) = \frac{\kappa_0(\beta)^{-1} - \kappa_0(\beta)}{\kappa_0(\beta)^{-1} + \kappa_0(\beta)}(X_c - I + 5D(\beta)) = \frac{\sqrt{5}}{3}(X_c - I + 5D) + O(\beta_c - \beta) \tag{4.12}$$

with the same D as in (3.6) and $\|O(\beta_c - \beta)\|/(\beta_c - \beta)$ bounded.

The natural small parameter and normalisation of m_i are

$$t = \left(\frac{\sqrt{5}}{3}(\beta_c - \beta)\right)^{1/2} \qquad x_i = m_i\left(\frac{\beta}{\beta_c - \beta}\right)^{1/2}. \tag{4.13}$$

In terms of the latter variables, (4.9) and (4.10) acquire the familiar form of (3.7), where

$$\boldsymbol{Q}_t(\boldsymbol{x}) = (X_c - I)\boldsymbol{x} + 5t^2\boldsymbol{x}(\boldsymbol{1} + \boldsymbol{xx}) - \Delta \cdot P\boldsymbol{x} + O(t^2) \cdot P\boldsymbol{1}. \tag{4.14}$$

Again we have included into the $O(t^2)$ term part of the last term of (4.11) and $t^2 x_1 \cdot P(X_c - I)(\boldsymbol{xx})$ which will not affect the behaviour of $\boldsymbol{x}^0(t)$ and will leave an $\boldsymbol{R}_t$ of the right order of magnitude (this can be checked using the analysis below in the same way as in § 4.1). The relevant map in solving the equation $\boldsymbol{Q}_t(\boldsymbol{x}) = \boldsymbol{0}$ is now

$$T_t^+\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ 2y - x + 5t^2 y(1 + y^2) \end{pmatrix}. \tag{4.15}$$

T_t^+ has a unique fixed point, $\binom{0}{0}$, and its tangent map at the fixed point has eigenvalues λ_t^+, $1/\lambda_t^+$, where $\lambda_t^+ = 1 - \sqrt{5}\,t + O(t^2)$.

Lemma 4. The stable manifold of the fixed point of T_t^+ is the graph of a strictly increasing, contractive, antisymmetric C^∞-function, $\varphi_t^+ : R \to R$ which is concave on R_+. Uniformly on compacts

$$\lim_{t \downarrow 0} t^{-1}[\varphi_t^+(x) - x] = h^+(x) \equiv -\sqrt{5}x(1 + \tfrac{1}{2}x^2)^{1/2}. \tag{4.16}$$

The proof of this result is sketched in appendix 1. It implies that the equation $\boldsymbol{Q}_t(\boldsymbol{x})=\boldsymbol{0}$ has a unique positive solution, $\boldsymbol{x}^0(t)$, such that $\lim_{i\to\infty} x^0(t)_i=0$, namely $x^0(t)_i=\varphi_t^+(x^0(t)_{i-1})$, and $x^0(t)_1$ determined from the first equation (4.14) as the solution of

$$\varphi_t^+(x)=x-(\Delta-\Delta_c)x+\mathrm{O}(t^2x). \tag{4.17}$$

We have used here the known values of Δ_c, $(X_c-I)_{11}$, cf (2.22) and (3.2). Replacing here the asymptotics (4.16) of φ_t^+, we obtain that, for $t\downarrow 0$ and $\Delta-\Delta_c=\delta t$ with $\delta>\delta_s=\sqrt{5}$,

$$\lim_{t\downarrow 0} x_i^0(t)=\tilde{x}_1^+(\delta)=[\tfrac{2}{5}(\delta^2-5)]^{1/2} \qquad i=1,2,\ldots \tag{4.18}$$

$$\lim_{t\downarrow 0, i\to\infty, it\to z} x_i^0(t)=\mu^+(z,\delta) \tag{4.19}$$

where $\mu^+(\cdot,\delta)$ is the solution of the Cauchy problem

$$\mathrm{d}\mu/\mathrm{d}z=h^+(\mu) \qquad \mu(0)=\tilde{x}_1^+(\delta) \tag{4.20}$$

i.e. $\mu^+(z,\delta)=\sqrt{2}/\sinh\sqrt{5}(z+z_0^+(\delta))$, $z_0^+(\delta)=(1/\sqrt{5})\sinh^{-1}(\sqrt{2}/\tilde{x}_1^+(\delta))$.

We mention a few differences in the proof that (4.19) and (4.20) hold true for the solution of the full equation $\boldsymbol{Q}_t(\boldsymbol{x})=t^4\boldsymbol{x}\boldsymbol{R}_t(\boldsymbol{x})$. The space B_t is defined as before but with $p_t=\varphi_t^{+\prime}(0)=\lambda_t^+$. Then, the construction of $\boldsymbol{x}^0(t)$ by iterating φ_t^+ on $\tilde{x}_1^+(\delta)$ ensures that $\boldsymbol{x}^0(t)\in B_t$ and $\|\boldsymbol{x}^0(t)\|_t$ is bounded for $t\downarrow 0$. Hence, also $t^{-2}\|\boldsymbol{v}(\boldsymbol{x}^0(t))\|_t$ is bounded, because essentially $t^{-2}\boldsymbol{v}(\boldsymbol{x}^0)=5\boldsymbol{x}^0\boldsymbol{x}^0$, which, in turn, ensures, via (3.24) and (3.23′) and the boundedness of

$$t^{-2}\sup_i\left|\frac{[(X_c-I)\boldsymbol{x}]_i}{x_i}\right|\leq t^{-2}\Big(\sup_i v_i+\sup_i(X_0(\beta)-X_c)_{ii}\Big)$$

that $\|\boldsymbol{R}_t(\boldsymbol{x}^0(t))\|_t$ is bounded.

In conclusion, around the special transition point, for $\beta\downarrow\beta_c$, $\Delta\to\Delta_c$ as well as for $\beta\uparrow\beta_c$, $\Delta\downarrow\Delta_c$, the solution has the critical behaviour and scales to the magnetisation profile predicted by the continuum Ginsburg–Landau theory, whereby $-1/\delta$ plays the role of the extrapolation length. In fact (4.7) and (4.20) are equivalent to the (second-order) Ginsburg–Landau equation supplemented with the appropriate boundary condition (compare with (3.4) of Lubensky and Rubin (1975)). Also, it is not hard to see that the limit of $t^2Q_t'(\boldsymbol{x}^0)^{-1}$ derived in appendix 2 gives the exact scaled form of the susceptibility matrix of the model.

It is somewhat more delicate to derive the corresponding result when (β,Δ) approaches (β_c,Δ_c) along the extraordinary transition line, i.e. $\beta=\beta_c$, $\Delta\downarrow\Delta_c$, and we shall confine ourselves to a few general remarks. In this regime, the only small parameter left is $\Delta-\Delta_c$. When trying to approximate the solution of (3.1) by an orbit of an area preserving map, it turns out that the latter does not change when Δ approaches Δ_c and has only one *degenerate* fixed point, the stable manifold of which has an asymptotics $\varphi(m)\sim m-cm^2$ around $m=0$. The only consequence of Δ approaching Δ_c is the fact that the initial point of the orbit moves towards the origin, $m_1\sim\Delta-\Delta_c$. Thus, in first approximation, $m_i=\varphi^{\circ i}(m_1)$, which correlated with the asymptotics of φ at 0, shows that m_i/m_1 scales again to the magnetisation profile predicted by the continuum theory: $\lim_{i\to\infty, i(\Delta-\Delta_c)\to z} m_i/m_1=(1+z)^{-1}$ (compare with (7.4) of Lubensky and Rubin (1975)).

Appendix 1. The stable manifold of the relevant fixed points

We present an easy *ad hoc* way of constructing φ_t. We start with the case $\beta > \beta_c$ and prove lemma 1.

Let $\mathscr{C}$ denote all functions $\varphi : [0, \infty) \to [0, \infty)$ which are non-decreasing, concave and satisfy $\varphi(1) = 1$. For $\varphi \in \mathscr{C}$, define $\mathscr{T}_t \varphi \in \mathscr{C}$ by

$$T_t \begin{pmatrix} x \\ \mathscr{T}_t \varphi(x) \end{pmatrix} = \begin{pmatrix} x' \\ \varphi(x') \end{pmatrix} \qquad \text{for some } x' \geq 0 \tag{A1.1}$$

where T_t is given by (3.12), i.e. the graph of $\mathscr{T}_t \varphi$ is in the image through T_t^{-1} of the graph of φ. Otherwise $\mathscr{T}_t \varphi$ is the inverse of the function $\psi(x') = 2x' - 5t^2 x'(1 - x'^2) - \varphi(x')$ restricted to $[\psi^{-1}(0), \infty)$, where $\psi^{-1}(0) \geq 0$ is the largest root of ψ (as ψ is strictly convex and $\psi(0) = -\varphi(0) \leq 0$, it has at most two roots and is strictly increasing beyond the larger non-negative root).

φ_t will be a fixed point of $\mathscr{T}_t$. By the general theory (Hirsch and Pugh 1970), if $\mathscr{T}_t$ has a fixed point in $\mathscr{C}$, this is unique and C^∞. Thus, it is sufficient to find a fixed point in $\mathscr{C}$. To this aim, start with $\varphi_0(x) \equiv 1$ and iterate $\mathscr{T}_t$ on φ_0. With $\psi_k(x') = 2x' - 5t^2 x'(1 - x'^2) - \varphi_k(x')$, we have $\varphi_{k+1} = \mathscr{T}_t \varphi_k = (\psi_k|_{[\psi_k^{-1}(0), \infty)})^{-1}$. We assert that φ_k are contractive and $\varphi_k(x)$ is monotonously decreasing (increasing) with k for fixed $x < 1$ ($x > 1$, respectively). We proceed by induction. $\psi_0(0) = -1$, $\psi_0(1) = 1$ and the convexity of ψ_0 imply $\psi_0^{-1}(0) < 1$ and $\psi_0'(\psi_0^{-1}(0)) > 1$, whereupon the assertion follows for $k = 1$. Then, if the assertion is true up to k, we have $\psi_k(x') \gtreqless \psi_{k-1}(x')$ for $x' \lesseqgtr 1$. Therefore $\psi_k(0) = -\varphi_k(0) = -\psi_{k-1}^{-1}(0) < 0$ and $\psi_k(\psi_{k-1}^{-1}(0)) > \psi_{k-1}(\psi_{k-1}^{-1}(0)) = 0$. As above, this implies $\psi_k^{-1}(0) < \psi_{k-1}^{-1}(0)$ and $\psi_k'(\psi_{k-1}^{-1}(0)) > 1$ which proves the claim for $k+1$.

Thus, $\varphi_t = \lim_{k \to \infty} \varphi_k$ exists pointwise and is a contraction belonging to $\mathscr{C}$. Also, ψ_k converge on $[0, \infty)$ to a convex function, which we denote, by a slight abuse, φ_t^{-1}. Explicitly $\mathscr{T}_t \varphi_t = \varphi_t$ means that φ_t satisfies the functional equation:

$$2\varphi_t(x) - \varphi_t \circ \varphi_t(x) - x = 5t^2 \varphi_t(x)[1 - \varphi_t(x)^2]. \tag{A1.2}$$

We shall prove (3.15) using (A1.2). To this aim, define

$$h_t(x) = t^{-1}[\varphi_t(x) - x] \tag{A1.3}$$

in terms of which the LHS of (A1.2) is written as $t[h_t(x) - h_t \circ \varphi_t(x)]$. h_t is decreasing, concave and $h_t(1) = 0$. Therefore, for $x < 1$, we have

$$h_t \circ \varphi_t^{-1}(x) > h_t(x) > h_t \circ \varphi_t(x) > 0 \tag{A1.4}$$

$$-t h_t(x) h_t'(x) < h_t(x) - h_t \circ \varphi_t(x) < -t h_t(x) h_t' \circ \varphi_t(x). \tag{A1.5}$$

Inserting (A1.2) for the middle part of (A1.5), multiplying by $\varphi_t'(x)$ and integrating from $\varphi_t^{-1}(x)$ to 1, we obtain

$$-\int_{\varphi_t^{-1}(x)}^1 h_t(y) h_t'(y) \varphi_t'(y)\, dy < 5 \int_x^1 y(1 - y^2)\, dy < -\int_{\varphi_t^{-1}(x)}^1 h_t(y)(h_t \circ \varphi_t)'(y)\, dy. \tag{A1.6}$$

Now, $\varphi_t'(x) > \varphi_t'(1) = \lambda_t$ and $h_t(x) < (h_t \circ \varphi_t(x))/(1 + t h_t' \circ \varphi_t(x)) < (h_t \circ \varphi_t(x))/\lambda_t$ (use the second inequality (A1.5) and concavity), so (A1.6) becomes

$$\lambda_t [h_t \circ \varphi_t^{-1}(x)]^2 < \tfrac{5}{2}(1 - x^2)^2 < \lambda_t^{-1}[h_t(x)]^2. \tag{A1.7}$$

Using (A1.7), (A1.4) and $\lambda_t \to 1$ for $t \to 0$, we obtain (3.15) for $x < 1$. For $x > 1$, the opposite bracketing holds and the same argument works.

We consider next the case $\beta < \beta_c$ and sketch a proof of lemma 4. Let $\mathscr{C}^+$ denote all antisymmetric functions $\varphi: R \to R$ which are positive, non-decreasing, contracting and concave on $[0, \infty)$. Define as before $\mathscr{T}_t^+ : \mathscr{C}^+ \to \mathscr{C}^+$ by

$$T_t^+ \begin{pmatrix} x \\ \mathscr{T}_t^+ \varphi(x) \end{pmatrix} = \begin{pmatrix} x' \\ \varphi(x') \end{pmatrix} \qquad \text{for some } x' \in R. \tag{A1.8}$$

Explicitly, $\mathscr{T}_t^+ \varphi = \psi^{-1}$, where $\psi(x) = 2x + 5t^2 x(1+x^2) - \varphi(x)$. When starting with $\varphi_0 = 0$ and defining $\varphi_k = \mathscr{T}_t^+ \varphi_{k-1}$ $(k \geqslant 1)$, one finds that $\varphi_k(x)$ increases with k for every fixed $x > 0$. Therefore $\varphi_t^+(x) = \lim_{k\to\infty} \varphi_k(x)$ exists and $\varphi_t^+ \in \mathscr{C}^+$ is a fixed point of $\mathscr{T}_t^+$, i.e. a solution of the functional equation

$$2\varphi_t^+(x) - \varphi_t^+ \circ \varphi_t^+(x) - x = -5t^2 \varphi_t^+(x)(1 + \varphi_t^+(x)^2). \tag{A1.9}$$

The critical behaviour of φ_t^+, (4.16), is obtained again by bracketing the LHS of (A1.9), exploiting concavity. Namely, defining h_t^+ in terms of φ_t^+ as in (A1.3), we have

$$h_t^+ \circ (\varphi_t^+)^{-1}(x) < h_t^+(x) < h_t^+ \circ \varphi_t^+(x) < 0 \tag{A1.10}$$

$$-th_t^+(x) h_t^{+\prime}(x) < h_t^+(x) - h_t^+ \circ \varphi_t^+(x) < -th_t^+(x) h_t^{+\prime} \circ \varphi_t^+(x) \tag{A1.11}$$

from which, as before,

$$\begin{aligned} -\int_0^{(\varphi_t^+)^{-1}(x)} h_t^+(y) h_t^{+\prime}(y) \varphi_t^{+\prime}(y)\, \mathrm{d}y &< -\int_0^x 5y(1+y^2)\, \mathrm{d}y \\ &< -\int_0^{(\varphi_t^+)^{-1}(x)} h_t^+(y) (h_t^+ \circ \varphi_t^+)'(y)\, \mathrm{d}y. \end{aligned} \tag{A1.12}$$

The LHS of (A1.12) is larger than $-\lambda_t^+ h_t^+ \circ (\varphi_t^+)^{-1}(x)/2$, because $h_t^+(y) h_t^{+\prime}(y) > 0$ and $\varphi_t^{+\prime}(y) < \lambda_t^+$ for $y > 0$; likewise, $h_t^+(y)$ can be bounded by $h_t^+ \circ \varphi_t^+(y)$ in the RHS which gives the upper bound $-\frac{1}{2} h_t^+(y)^2$. Thus lemma 4 is proved.

Appendix 2

We shall sketch here a proof of the fact that $t^2 Q_t'(x^0(t))^{-1}$ is a bounded operator in B_t for all t less than some t_0 and its norm is bounded uniformly in $0 < t \leqslant t_0$. The proof depends of course on the scaling regime we approach.

We start with the ordinary transition as studied in § 3. For $\beta > \beta_c$, $\Delta = 0$, we have

$$t^{-2} Q_t'(x^0) = L_t - \hat{W}(x^0) \tag{A2.1}$$

where

$$L_t = t^{-2}(X_c - I) + 10I \tag{A2.2}$$

$$W(x^0)_i = 15[1 - (x_i^0)^2] + [\tfrac{1}{2}(3\sqrt{5} - 5) + 3(\sqrt{5} + 8)(x_1^0)^2]\delta_{i1}. \tag{A2.3}$$

In (A2.1), L_t is minus a discrete Laplacian shifted upwards such that now the perturbing potential W_i vanishes for $i \to \infty$. More precisely, from corollary 2, we have, for some t_0, $C > 0$, that

$$\| \hat{W}(x^0) \|_t \leqslant C \qquad \forall t \leqslant t_0. \tag{A2.4}$$

Our starting point is the perturbation formula

$$t^2 Q_t'(x^0)^{-1} = L_t^{-1} + L_t^{-1} \hat{W}(I - L_t^{-1} \hat{W})^{-1} L_t^{-1}. \tag{A2.5}$$

The matrix elements of L_t^{-1} are calculated explicitly as

$$(L_t^{-1})_{ij}=\frac{t^2}{\lambda_t^{-1}-\lambda_t}\left(\lambda_t^{|i-j|}-\frac{a_t\lambda_t^{-2}}{a_t+\lambda_t^{-1}-\lambda_t}\lambda_t^{i+j}\right)\leqslant\frac{t^2}{\lambda_t^{-1}-\lambda_t}\lambda_t^{|i-j|}\qquad(\text{A2.6})$$

where $a_t=\frac{1}{2}(1+\sqrt{5})+10t^2-\lambda_t^{-1}$. Using (A2.6) one can easily estimate the norm of L_t^{-1} as an operator in B_t

$$\|L_t^{-1}y\|_t\leqslant\|y\|_t\frac{t^2}{\lambda_t^{-1}-\lambda_t}\sup_i\left(p_t^{-i}\sum_j\lambda_t^{|i-j|}p_t^j\right)$$

$$\leqslant\|y\|_t\frac{t^2}{\lambda_t^{-1}-\lambda_t}\left(\frac{1}{p_t-\lambda_t}+\frac{1}{1-p_t\lambda_t}\right).$$

Equations (3.14) and (3.21) imply that, for some $K_1>0$,

$$\|L_t^{-1}\|_t\leqslant K_1.\qquad(\text{A2.7})$$

To control the second term in (A2.5), it will be convenient to look at $(I-L_t^{-1}\hat{W})^{-1}$ as an operator in l^2, where the estimate is easier; as a consequence, we need to bound the norms of the two factors sandwiching it in (A2.5), namely of I and $L_t^{-1}\hat{W}$, as operators from B_t to l^2 and from l^2 to B_t, respectively. We have

$$\|y\|_{l^2}^2=\sum_i|y_i|^2\leqslant\|y\|_t^2\sum_i p_t^{2i}\leqslant(K_2^2/t)\|y\|_t^2\qquad(\text{A2.8})$$

and

$$\|L_t^{-1}\hat{W}y\|_t\leqslant\|y\|_{l^2}\sup_i\left[p_t^{-i}\left(\sum_j(L_t^{-1})_{ij}^2W_j^2\right)^{1/2}\right]$$

$$\leqslant\|y\|_{l^2}\frac{Ct^2}{\lambda_t^{-1}-\lambda_t}\sup_i\left[p_t^{-i}\left(\sum_j\lambda_t^{2|i-j|}p_t^{2j}\right)^{1/2}\right]\qquad(\text{A2.9})$$

$$\leqslant\|y\|_{l^2}\frac{Ct^2}{\lambda_t^{-1}-\lambda_t}\left(\frac{1}{p_t^2-\lambda_t^2}+\frac{1}{1-\lambda_t^2p_t^2}\right)^{1/2}\leqslant K_3t^{1/2}\|y\|_{l^2}$$

where we used the Schwarz inequality, (A2.4), (A2.6) and again (3.14), (3.21). Collecting the estimates, we have

$$\|t^2Q_t'(x^0)^{-1}\|_t\leqslant K_1+K_1K_2K_3(1-\|L_t^{-1}\hat{W}\|_{l^2})^{-1}\qquad(\text{A2.10})$$

so we are left with proving that $\|L_t^{-1}\hat{W}\|_{l^2}\leqslant r<1$ with r independent of t, for t sufficiently small. This will be done by going to a 'continuum limit'. Let $U_t:l^2\to L^2(0,\infty)$ be defined by

$$(U_t\xi)(z)=t^{-1/2}\xi_{[z/t]}\qquad z\geqslant 0\qquad(\text{A2.11})$$

where $[z]$ denotes the least integer larger than z. U_t is isometric and its adjoint $U_t^*:L^2(0,\infty)\to l^2$ is given by

$$(U_t^*f)_i=t^{-1/2}\int_{(i-1)t}^{it}f(z)\,\mathrm{d}z\qquad i=1,2,\ldots.\qquad(\text{A2.12})$$

The identity of l^2 has the representation $I=U_t^*U_t$. Thus

$$\|L_t^{-1}\hat{W}\|_{l^2}=\|U_t^*U_tL_t^{-1}\hat{W}U_t^*U_t\|_{l^2}\leqslant\|(U_tL_t^{-1}U_t^*)(U_t\hat{W}U_t^*)\|_{L^2(0,\infty)}.$$

Now, if A is an operator in l^2 of the matrix (A_{ij}), then $U_tAU_t^*$ is an integral operator of kernel $(U_tAU_t^*)(z,z') = t^{-1}A_{[z/t],[z'/t]}$. Using this and (A2.6), one can verify that $U_tL_t^{-1}U_t^*$ converges in norm to the integral operator of kernel $(1/2\sqrt{10})\times [\exp(-\sqrt{10}|z-z'|)-\exp(-\sqrt{10}(z+z'))]$, i.e. to $(-\mathrm{d}^2/\mathrm{d}z^2+10)^{-1}$, where $\mathrm{d}^2/\mathrm{d}z^2$ is supplemented with zero Dirichlet boundary conditions at $z=0$ (which makes it a self-adjoint operator in L^2). Also, $U_t\hat{W}U_t^*$ converges in norm to $\hat{w}$, the multiplication by $w(z)=15/\cosh^2(\frac{1}{2}\sqrt{10}z)$, as follows from corollaries 1 and 2, which control the uniform convergence on compacts and the uniform asymptotics, respectively. Finally

$$\|(-\mathrm{d}^2/\mathrm{d}z^2+10)^{-1}\hat{w}\|_{L^2(0,\infty)}<1 \tag{A2.13}$$

because $-\mathrm{d}^2/\mathrm{d}z^2+10-\hat{w}$ is strictly positive definite, as can be seen by calculating explicitly its inverse (see below). Let us remark that, in fact, we proved that $t^2U_tQ_t'(\boldsymbol{x}^0)^{-1}U_t^*$ converges in norm on $L^2(0,\infty)$ to the inverse of $-\mathrm{d}^2/\mathrm{d}z^2+10-\hat{w}$ supplemented with zero Dirichlet boundary conditions at $z=0$. The kernel of the latter operator can be obtained by integrating the differential equation $[-\mathrm{d}^2/\mathrm{d}z^2+10-w(z)]\psi(z)=0$. Let

$$\begin{aligned} y(z) &= 1/\cosh^2(\tfrac{1}{2}\sqrt{10}z)\\ u(z) &= \tfrac{3}{8}z+(1/2\sqrt{10})\sinh(\sqrt{10}z)+(1/16\sqrt{10})\sinh(2\sqrt{10}z). \end{aligned} \tag{A2.14}$$

Then, $y(z)$ is the solution vanishing at $+\infty$, and $y(z)u(z)$ is the solution satisfying the boundary condition $u(0)=0$, so the resolvent kernel is $y(z)u(z_m)y(z')$, where $z_m=\min(z,z')$. In particular, we proved

$$\lim_{\substack{i,j\to\infty,t\to 0\\ it\to z,jt\to z'}} t[Q_t'(\boldsymbol{x}^0)^{-1}]_{ij} = y(z)u(z_m)y(z'). \tag{A2.15}$$

We conclude this case with the remark that, for $\Delta<\Delta_c$ fixed, the result and its proof do not change, because $\boldsymbol{W}$ interpolates the same function, $w(z)$, and $U_tL_t^{-1}U_t^*$ converges to the same operator in $L^2(0,\infty)$ (because $\lim_{t\to 0} a_t=\frac{1}{2}(1+\sqrt{5})-\Delta-1>0$, so, in (A2.6), $\lim_{t\to 0} a_t\lambda_t^{-2}/(a_t+\lambda_t^{-1}-\lambda_t)=1$).

The situation changes when $\Delta\to\Delta_c$, $\beta\downarrow\beta_c$, as considered in § 4.1. In this case, essentially

$$t^{-2}Q_t'(\boldsymbol{x}^0)=L_t^0-\hat{W}_\delta(\boldsymbol{x}^0)-(\delta/t)P\equiv\Lambda_{\delta,t}-(\delta/t)P \tag{A2.16}$$

where

$$L_t^0=t^{-2}(X_c-I-\Delta_cP)+10I \tag{A2.17}$$

$$\lim_{i\to\infty,t\to 0,it\to z}(W_\delta)_i=w_\delta(z)=1-\mu(z,\delta)^2$$

$$=\begin{cases} 15/\cosh^2\dfrac{\sqrt{10}}{2}(z+z_0(\delta)) & \text{for } \delta<0\\ 0 & \text{for } \delta=0\\ -15/\sinh^2\dfrac{\sqrt{10}}{2}(z+z_0(\delta)) & \text{for } \delta>0. \end{cases} \tag{A2.18}$$

We omit all details related with going from B_t to l^2 and back, and sketch only the core of the argument, which is the analysis in l^2. We consider first the rank-one boundary perturbation, which is strongly felt in this case. We have

$$t^2Q_t'(\boldsymbol{x}^0)^{-1}=\Lambda_{\delta,t}^{-1}+(\delta/t)\Lambda_{\delta,t}^{-1}P\Lambda_{\delta,t}^{-1}[1-(\delta/t)(\Lambda_{\delta,t}^{-1})_{11}]^{-1} \tag{A2.19}$$

so it will be sufficient to show that $\Lambda_{\delta,t}^{-1}$, $t^{-1}\Lambda_{\delta,t}^{-1}P\Lambda_{\delta,t}^{-1}$ are uniformly bounded and $\lim \delta t^{-1}(\Lambda_{\delta,t}^{-1})_{11} < 1$. To this aim, we apply in turn to $\Lambda_{\delta,t}^{-1}$ a perturbation formula like (A2.5). The main difference is that now L_t^{0-1} is given by (A2.6) with $a_t = -t\sqrt{10} + O(t^2)$, so

$$\lim_{t\to 0}(U_t L_t^{0-1} U_t^*)(z, z') = \frac{1}{2\sqrt{10}}[\exp(-\sqrt{10}|z-z'|) + \exp(-\sqrt{10}(z+z'))] \tag{A2.20}$$

i.e. $U_t L_t^{0-1} U_t^*$ converges to the resolvent of $-\mathrm{d}^2/\mathrm{d}z^2$ with Neumann boundary conditions at 0 (zero derivative) on $L^2(0,\infty)$. From (A2.18) and (A2.20) we obtain as before that $U_t\Lambda_{\delta,t}^{-1}U_t^*$ converges in norm on $L^2(0,\infty)$ to the inverse of $-\mathrm{d}^2/\mathrm{d}z^2 + 10 - \hat{w}_\delta$ supplemented with Neumann boundary conditions at 0. We have

$$\lim_{t\to 0}(U_t\Lambda_{\delta,t}^{-1}U_t^*)(z, z') = y_\delta(z)u_\delta(z_m)y_\delta(z') \tag{A2.21}$$

where y_δ, u_δ can again be calculated in analytic form (cf, e.g., Lubensky and Rubin 1975). We do not reproduce these rather cumbersome formulae. They show that the limit operator (A2.21) is positive and bounded and that, moreover, the rank-one operator in (A2.19) converges

$$\lim_{t\to 0} t^{-1}(U_t\Lambda_{\delta,t}^{-1}P\Lambda_{\delta,t}^{-1}U_t^*)(z, z') = y_\delta(z)y_\delta(z')(y_\delta(0)u_\delta(0))^2$$

$$= y_\delta(z)y_\delta(z')/10\tilde{x}_1(\delta)^2(1-\tilde{x}_1(\delta)^2)^2 \tag{A2.22}$$

$$\lim_{t\to 0} t^{-1}(\Lambda_{\delta,t}^{-1})_{11} = y_\delta(0)^2u_\delta(0) = 1/\sqrt{10}\tilde{x}_1(\delta). \tag{A2.23}$$

Remembering (4.5), one sees that all the terms in (A2.19) are uniformly bounded and

$$\lim_{t\to 0} t^2(U_tQ_t'(\boldsymbol{x}^0)^{-1}U_t^*)(z, z') = y_\delta(z)u_\delta(z_m)y_\delta(z')$$

$$+\delta y_\delta(z)y_\delta(z')/10\tilde{x}_1(\delta)^2(1-\tilde{x}_1(\delta)^2)^2(1-\delta/\sqrt{10}\tilde{x}_1(\delta)). \tag{A2.24}$$

The RHS of (A2.24) is the kernel of the inverse of $-\mathrm{d}^2/\mathrm{d}z^2 + 10 - \hat{w}_\delta$ on $L^2(0,\infty)$ supplemented with the boundary condition

$$\psi'(0) = -\delta\psi(0). \tag{A2.25}$$

Finally, for the case considered in § 4.2, $\beta < \beta_c$ with $t = ((\sqrt{5}/3)(\beta_c - \beta))^{1/2}$, $\Delta - \Delta_c = \delta t$ $(\delta > \sqrt{5})$, we have (omitting irrelevant higher-order terms)

$$t^{-2}Q_t'(\boldsymbol{x}^0) = L_t^0 + \hat{W}_\delta^+(\boldsymbol{x}^0) - (\delta/t)P \equiv \Lambda_{\delta,t}^+ - (\delta/t)P \tag{A2.26}$$

where $L_t^0 = t^{-2}(X_c - I - \Delta_c P) + 5I$, $\hat{W}_\delta^+(\boldsymbol{x}^0)_i = 15(x_i^0)^2$ and

$$\lim_{i\to\infty, t\to 0, it\to z} W_\delta^+(\boldsymbol{x}^0)_i = w_\delta^+(z) = 15\mu^+(z,\delta) = 30/\sinh\sqrt{5}(z + z_0(\delta)). \tag{A2.27}$$

Following the same procedure, we obtain the convergence of $(\Lambda_{\delta,t}^+)^{-1}$ (the analogue of (A2.21)). Also, (A2.22)–(A2.24) have their counterparts here, which, when introduced into the perturbation (A2.19), show that $U_t t^2 Q_t'(\boldsymbol{x}^0)^{-1}U_t^*$ converges in norm in $L^2(0,\infty)$ to the inverse of $-\mathrm{d}^2/\mathrm{d}z^2 + 5 + \hat{w}_\delta$ supplemented with the boundary condition (A2.25).

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